Modern University
For Information and Technology
Department: Basic Since
Specialization: Mech. &civil & Bio& Comm. and control Dep.

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Final Term Examination

Subject: Mathematic III **Code:** CENG 203, MENG208, ELTE201,CONT201, COM201
Examiner: Dr. Fathi Abdessalam **Time Allowed: 2 hours**

Number of Pages: 1	Number of Questions: 5	Attempt all Questions

<u>Question 1,</u> (8 Marks)

(a) Find the first and the second partial derivative for $f(x,y)=(x^3-y^2)^5$

<u>Answer</u>

$$\frac{\partial f(x,y)}{\partial x}=5(x^3-y^2)^4(3x^2)=15x^2(x^3-y^2)^4$$

$$\frac{\partial f(x,y)}{\partial y}=5(x^3-y^2)^4(-2y)=-10y(x^3-y^2)^4$$

$$\frac{\partial^2 f(x,y)}{\partial x^2}=15x^2(4)(x^3-y^2)^3(3x^2)+30x(x^3-y^2)^4$$

$$\frac{\partial^2 f(x,y)}{\partial y^2}=-10(x^3-y^2)^4-10y(4)(x^3-y^2)^3(-2y)$$

$$\frac{\partial^2 f(x,y)}{\partial y\,\partial x}=-10y(4)(x^3-y^2)^4(3x^2)$$

(b) If $w=\ln(x^2+y^2)$ Show that $x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}=2$

<u>Answer</u>

$$w=\ln(x^2+y^2)$$

$$\frac{\partial w}{\partial x}=\frac{2x}{(x^2+y^2)} \qquad \Rightarrow x\frac{\partial w}{\partial x}=\frac{2x^2}{(x^2+y^2)}$$

$$\frac{\partial w}{\partial y}=\frac{2y}{(x^2+y^2)} \qquad \Rightarrow y\frac{\partial w}{\partial y}=\frac{2y^2}{(x^2+y^2)}$$

$$\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}=\frac{2y^2}{(x^2+y^2)}+\frac{2x^2}{(x^2+y^2)}=\frac{2(x^2+y^2)}{(x^2+y^2)}=2$$

Question 2, (8 Marks)

(a) $x\sqrt{1+y^2}dx + y\sqrt{1+x^2}dy = 0$

Answer

$x\sqrt{1+y^2}dx + y\sqrt{1+x^2}dy = 0$ divided by $\sqrt{1+y^2}\sqrt{1+x^2}$ we get

$\frac{x}{\sqrt{1+x^2}}dx + \frac{y}{\sqrt{1+y^2}}dy = 0$ then integrate $\frac{1}{2}\int \frac{2x}{\sqrt{1+x^2}}dx + \int \frac{1}{2}\frac{2y}{\sqrt{1+y^2}}dy = 0$

$\therefore \frac{1}{2}\left(2\sqrt{1+x^2}\right) + \frac{1}{2}\left(2\sqrt{1+y^2}\right) = c \qquad \Rightarrow \qquad \boxed{\left(\sqrt{1+x^2}\right) + \left(\sqrt{1+y^2}\right) = c}$

(b) $(xy - x^2)dy - y^2dx = 0$

Answer

$M(x,y), N(x,y)$ are homogeneous of the same degree(second degree)

let $\boldsymbol{y = ux}$ $\therefore dy = udx + xdu$

Substitute in the differential equation we have

$(x^2u - x^2)(udx + xdu) - x^2u^2dx = 0$

$x^2(u-1)(udx + xdu) - x^2u^2dx = 0$

Divided by $\boldsymbol{x^2}$ we have $(u-1)(udx + xdu) - u^2dx = 0$

$(u-1)udx + (u-1)xdu - u^2dx = 0$

$\left[(u-1)u - u^2\right]dx + (u-1)xdu = 0$

$-udx + (u-1)xdu = 0$ separate the variables $-\frac{dx}{x} + \frac{(u-1)}{u}du = 0$

$-\frac{dx}{x} + (1 - \frac{1}{u})du = 0$ by integration we have $-\ln x + u - \ln u + \ln C = 0$

$\ln\frac{xu}{C} = u \qquad \Rightarrow \qquad \boxed{y = Ce^{y/x}}$

Another Solution

$(xy - x^2)dy - y^2dx = 0$

$(xydy - x^2dy - y^2dx = 0$

$(xydy - y^2dx - x^2dy = 0$

$y\left(xdy-ydx\right)-x^{2}dy=0$ divide the equation by $x^{2}y$ we get

$$\frac{\left(xdy-ydx\right)}{x^{2}}-\frac{dy}{y}=0 \quad \Rightarrow \quad d\left(\frac{y}{x}\right)-\frac{dy}{y}=0$$ inteqrate

$$\int d\left(\frac{y}{x}\right)-\int\frac{dy}{y}=c$$

$$\frac{y}{x}-\ln y=\ln C$$

$$\frac{y}{x}=\ln C-\ln y=\ln Cy$$

$$Cy=e^{y/x} \quad \Rightarrow \quad \boxed{y=Ae^{y/x}}$$

<u>Question 3</u>, (8 Marks)

(a) $(D^{2}-3D+2)y=e^{3x}$

<u>Answer</u>

The characteristic equation is $(m-2)(m-1)=0$

$$m_1=2,\ m_2=1 \quad \Rightarrow y_c=Ae^{2x}+Be^{x}$$

$$y_P=\frac{1}{(D-2)(D-1)}e^{x}=\frac{1}{(D-1)}\left[\frac{1}{(D-2)}e^{x}\right]=\frac{1}{(D-1)}\left[\frac{1}{(1-2)}e^{x}\right]$$

$$=\frac{-1}{(D-1)}e^{x}\left[\text{use the formula } \frac{1}{(D-m)^{r}}e^{mx}=\frac{x^{r}}{r!}e^{mx}\right]=-xe^{x}$$

Then the general solution $\boxed{y_G=Ae^{2x}+Be^{x}-xe^{x}}$

(b) $(D^{2}+D-2)y=\sin x$

<u>Answer</u>

The characteristic equation is $(m+2)(m-1)=0$ with roots

$m_1=-2,\ m_2=1$ then $y_c=Ae^{-2x}+Be^{x}$

$$y=\frac{1}{(D^{2}+D-2)}\sin x=\frac{1}{(-1+D-2)}\sin x=\frac{1}{(D-3)}\sin x$$

$$=\frac{(D+3)}{(D-3)(D+3)}\sin x=\frac{(D+3)}{(D^2-9)}\sin x=\frac{(D+3)}{(-10)}\sin x=\frac{-1}{10}(\cos x+3\sin x)$$

then the general solution $\boxed{y_G=Ae^{-2x}+Be^{x}+\frac{-1}{10}(\cos x+3\sin x)}$

Question 4, (8 Marks)

(a) Find $\overline{\nabla}.\overline{F}$ and $\overline{\nabla}\times\overline{F}$ given that $\vec{F}=(2x^2y+z^2)\vec{i}+x^2\vec{j}+3x^2z^3\vec{k}$

Answer

$$\overline{\nabla}.\overline{F}=(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k}).\left[(2x^2y+z^2)\vec{i}+x^2\vec{j}+3x^2z^3\vec{k}\right]$$

$$\overline{\nabla}.\overline{F}=(\frac{\partial}{\partial x}(2x^2y+z^2)+\frac{\partial}{\partial y}x^2+\frac{\partial}{\partial z}3x^2z^3)=\boxed{4xy+0+9x^2z^2}$$

$$\overline{\nabla}.\overline{F}=(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k})\times\left[(2x^2y+z^2)\vec{i}+x^2\vec{j}+3x^2z^3\vec{k}\right]$$

$$=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ (2x^2y+z^2) & x^2 & 3x^2z^3\end{vmatrix}$$

$$=\left(\frac{\partial}{\partial y}3x^2z^3-\frac{\partial}{\partial z}x^2\right)\vec{i}+\left(\frac{\partial}{\partial z}(2x^2y+z^2)-\frac{\partial}{\partial x}3x^2z^3\right)\vec{j}$$

$$+\left(\frac{\partial}{\partial x}x^2-\frac{\partial}{\partial y}(2x^2y+z^2)\right)\vec{k}=\boxed{0\vec{i}+\left(2z-6xz^3\right)\vec{j}+\left(2x-2x^2\right)\vec{k}}$$

(b) Find $\oint\overline{F}.dr$ where $\vec{F}=(x-y)\vec{i}+(x+y)\vec{j}$ on the circle $x=2\cos\theta,\ y=2\sin\theta$

Answer

$$\int_C\vec{F}.d\vec{r}=\int_C\left[(x-y)\vec{i}+(x+y)\vec{j}\right].(dx\,\overline{i}+dy\,\overline{j})=\int_C[(x-y)\,dx+(x+y)dy]$$

since

$$x=2\cos\theta\quad\Rightarrow\quad dx=-2\sin\theta\,d\theta$$
$$y=2\sin\theta\quad\Rightarrow\quad dy=2\cos\theta\,d\theta$$

substitute in the integral we have

$$\int_C \vec{F}.d\vec{r} = \int_0^{2\pi} [2\cos\theta - 2\sin\theta](-2\sin\theta)d\theta + [2\cos\theta + 2\sin\theta](2\cos\theta)d\theta$$

$$\int_C \vec{F}.d\vec{r} = \int_0^{2\pi} \left[-4\cos\theta\sin\theta + 4\sin^2\theta + 4\cos^2\theta + 4\sin\theta\cos\theta\right]d\theta$$

$$\int_C \vec{F}.d\vec{r} = \int_0^{2\pi} \left[4\sin^2\theta + 4\cos^2\theta\right]d\theta = \int_0^{2\pi} 4d\theta = 4\theta_0^{2\pi} = 8\pi$$

<u>Another solution</u>

$$\int_C \vec{F}.d\vec{r} = \int_C [(x-y)\,dx + (x+y)dy] = \iint \left(\frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial x}(x-y)\right)dxdy$$

$$= \iint 2dxdy = 2(\text{area of the circle}) = 2(2^2\pi) = \boxed{8\pi}$$

<u>Question 5,</u> (8 Marks)

(a) Evaluate $\int_C \frac{z^2+5}{z-1}dz$ where $\boldsymbol{C}$ is the circle $z = 2e^{i\theta}$.

<u>Answer</u>

Since $\boldsymbol{z = 1}$ inside the circle $|\boldsymbol{z}| = \boldsymbol{2}$ and

$\frac{f(z)}{z-z_0} = \frac{z^2+5}{z-1}$ then $z_0 = 1, f(z) = z^2+5$

$f(z_0) = (1)^2 + 5 = 6$ by using Cauchy's integral Formula

$\oint_C \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0)$ then $\int_C \frac{z^2+5}{z-1}dz = (2\pi i)f(1) = \boxed{24\pi i}$

(b) Use ratio test to test the series $\sum_{n=1}^{\infty} \frac{3n+2}{5^n}$ for convergence.

<u>Answer</u>

$$a_n = \frac{3n+2}{5^n}, \quad a_{n+1} = \frac{3n+5}{5^{n+1}} \to \frac{a_{n+1}}{a_n} = \frac{3n+5}{5^{n+1}}.\frac{5^n}{3n+2} = \frac{1}{5}\left(\frac{3n+5}{3n+2}\right)$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{5}\left(\frac{3n+5}{3n+2}\right) = \frac{1}{5} < 1$$

The series convergence series
